Analyzing Social Connections as a Laplacian Matrix

Rowan Hennessy

Abstract

We examine the connectivity of a graph with nodes representing "trip leaders" and edges representing "leading a trip together", interpreting the social connectivity of such a graph for use in future trip leading pairings.

Introduction

In the summer of 2022, I led wilderness trips alongside seven other first-year male staff members at YMCA Camp Widjiwagan, a wilderness tripping camp based out of Ely, Minnesota. Each staff member was assigned trips, each of which could be solo-led (read: only one staff member) or co-lead (read: two staff members).

In Math 2210, we learned about the Fiedler set of a graph, and as a learning experience in algebraic connectivity of a graph, we'll formalize this simple data into a graph and examine the Laplacian Matrix and Fiedler Set of this simple dataset.

Network Description

We'll begin with formalizing our data. Below is a table of co-leaders and the leaders they co-led with, anonymized.

Name	Summer 2021 Co-Leads
v_1	v_3, v_5, v_6, v_8
v_2	v_5, v_6
v_3	v_1, v_4, v_5, v_6, v_8
v_4	v_3, v_8
v_5	v_1, v_2, v_3
v_6	v_1, v_2, v_3, v_7, v_8
v_7	v_6
v_8	v_1, v_3, v_4, v_6

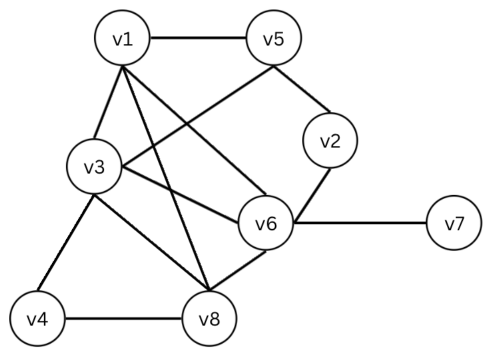
The discrepancy between trip leaders who led primarily solo-led trips becomes obvious here; v_7 only co-led a single trip, with v_6, while v_3 and v_6 both co-led five trips.

Formalizing, let our vertex set V be defined as

$$V = \{v_1, v_2, v_3, \dots v_8\}$$

with each vertex representing a single first-year staff member. Next, establish undirected edge set E as $\{(v_i, v_j) : \text{Trip Leaders } i \text{ and } j \text{ co-led a trip}\}$.

The vertex, edge pair (V, E) represents undirected graph G, pictured below:



forming the Laplacian Matrix

$$L = \begin{bmatrix} 4 & 0 & -1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 5 & -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 5 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & 0 & 4 \end{bmatrix}$$

where $L_{i,j},\ i \neq j = \begin{cases} 0 & \text{if trip leaders } i \text{ and } j \text{ did not co-lead, } i \neq j \\ -1 & \text{if trip leaders } i \text{ and } j \text{ did co lead, } i \neq j \\ C_i & i = j \end{cases}$

where C_i represents the total number of trips leader i led.

Fiedler Vector, Value, and Set

Visually, we see the graph is connected, and the Laplacian has nonzero diagonals, meaning the multiplicity of eigenvalue 0 is 1. Thus, the Fiedler value λ_2 must be positive, and no more addition of edges is necessary. Using Julia, we can find the Fiedler value to be

$$\lambda_2 = 0.8986544067634273$$

with corresponding Fiedler vector

$$\vec{\mu_2} = \begin{bmatrix} 0.1297 \\ 0.0758 \\ 0.1609 \\ 0.2893 \\ 0.17436 \\ -0.0909 \\ -0.8968 \\ 0.1577 \end{bmatrix}$$

Which gives Fiedler Parition

$$F = \{v_1, v_2, v_3, v_3, v_4, v_5, v_8\}$$

Analysis and Discussion

The most interesting aspect of the Fiedler partition is v_6 not being present. With leader 7 only leading a single trip, it intuitively makes sense for v_7 to not be present. However leader 6 led five trips, each with a distinct co-lead. Why is his node partitioned apart?

The obvious answer is that v_6 is unnecessary for the Fiedler Set's network. Even if v_6 were removed from G, as well as all edges with endpoints on v_6, the network would be unaffected, *except* for v_7 being disconnected. So, v_6 is essential for $V \backslash F$, but is unessential for F.

Regarding the two-way split's larger picture, the Fiedler Set is essentially the group of staff members who led the most trips, and thus worked with larger amounts of co-leads